Math 236: Some homework solutions

General comment: please justify your assertions; for instance, simply stating that a relation is symmetric will not earn full credit on an exam problem.

4a. (Note there is an implicit assumption here that a, b, c are *distinct* elements.) R is not transitive, since (a, b) and (b, a) are both in R, but (a, a) is not. Note that in the definition of transitivity, the statement must hold for all choices of three elements of A, which includes the case where some of the elements are equal.

4b. (i) Reflexive, since for all real x, $x - x = 0$, and 0 is rational. Symmetric, since $x - y = m/n$ for $m, n \in \mathbb{Z}$ with $n \neq 0$ implies $y - x = (-m)/n$, and $-m \in \mathbb{Z}$ because $m \in \mathbb{Z}$. Hence $y - x$ is rational. Transitive, since for all $x, y, z \in \mathbb{R}$, we have $x - z = (x - y) - (y - z)$, and the difference of two rational numbers is rational (you can see this by finding a common denominator for $\frac{m_1}{n_1} - \frac{m_2}{n_2}$).

4b. (ii) Not reflexive, since $1 - 1 = 0$ and 0 is not irrational, so $(1, 1)$ is not in R. Symmetric, since if $x - y$ is irrational, so must be $y - x$. You can prove this by contrapositive: suppose that $y - x$ was rational. Then by the symmetry of the relation in part 4b.(i), $x - y$ is rational. This is the contrapositive of the desired statement. Not transitive, since $(\sqrt{2}, 0)$ and $(0, \sqrt{2})$ are both in R, but $(\sqrt{2}, \sqrt{2})$ is not.

4b. (iii) Reflexive, since for all $x \in \mathbb{R}$, $|x - x| = 0 < 2$. Symmetric, since if $|x - y| < 2$, then $|y - x| = |-(x - y)| = |x - y|$, and the latter is < 2 by assumption. Not transitive, since $(3/2, 0) \in R$ and $(0, -3/2) \in R$, but $(3/2, -3/2) \notin R$.

6a. Not reflexive. Because U is not empty, there is a non-empty subset of U (for instance, U itself), and taking A to be this subset, we have $A \cap A \neq \emptyset$, so $(A, A) \notin R$. Symmetric, since $A \cap B = \emptyset$ implies $B \cap A = A \cap B = \emptyset$. Not transitive. Let A be a non-empty subset of U, and note that $(A, \emptyset) \in R$ and $(\emptyset, A) \in R$, but $(A, A) \notin R$.

6a. Not reflexive. Observe that $\emptyset \in P(A)$, but $(\emptyset, \emptyset) \notin R$. Symmetric, since $A \cap B \neq \emptyset$ implies $B \cap A = A \cap B \neq \emptyset$. Not transitive: let $U = \{1, 2, 3\}$, $A = \{1\}$, $B = \{1, 2\}$, and $C = \{2\}$. Then $(A, B) \in R$ and $(B, C) \in R$, but $(A, C) \notin R$.

8c. We prove that $R = A \times A$ (i.e., the biggest possible relation on A) is not anti-symmetric, and hence not a partial order. By assumption A has at least two elements, and thus we can take $a, b \in A$ with $a \neq b$ (this is where the proof breaks for sets A with one or zero elements.) Then (a, b) and (b, a) are both in R, but $a \neq b$, showing that R is not anti-symmetric.